

Noise Kernel and Stress Energy Bi-Tensor of Quantum Fields in Hot Flat Space and Gaussian Approximation in the Optical Schwarzschild Metric

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(Dated: umdpp 03-016, Submitted to Phys. Rev. D, September 17, 2002)

Continuing our investigation of the regularization of the noise kernel in curved spacetimes [N. G. Phillips and B. L. Hu, Phys. Rev. D **63**, 104001 (2001)] we adopt the modified point separation scheme for the class of optical spacetimes using the Gaussian approximation for the Green functions a la Bekenstein-Parker-Page. In the first example we derive the regularized noise kernel for a thermal field in flat space. It is useful for black hole nucleation considerations. In the second example of an optical Schwarzschild spacetime we obtain a finite expression for the noise kernel at the horizon and recover the hot flat space result at infinity. Knowledge of the noise kernel is essential for studying issues related to black hole horizon fluctuations and Hawking radiation backreaction. We show that the Gaussian approximated Green function which works surprisingly well for the stress tensor at the Schwarzschild horizon produces significant error in the noise kernel there. We identify the failure as occurring at the fourth covariant derivative order.

PACS numbers: 04.62.+v

I. INTRODUCTION

The noise kernel is the vacuum expectation value of the stress-energy bi-tensor for a quantum field. In curved spacetime field theory [1, 2, 3, 4] it plays a role in stochastic semiclassical gravity [5, 6, 7, 8, 9, 10] similar to the expectation value of the stress-energy tensor in semiclassical gravity [11]. The noise kernel being a measure of the fluctuations of the stress tensor of quantum fields enters in a great variety of issues and problems ranging from the validity of semiclassical gravity to spacetime foams, from structure formation in the early universe to fluctuations of the black hole horizon and trans-Planckian physics (for a review of this subject see [10]). Noise kernel in hot flat space, one of the two examples considered here, has been studied by Campos and Hu [7]. It is useful for performing a nonequilibrium quantum field theoretical analysis of the black hole nucleation problems beyond Euclidean thermodynamics [12, 13].

In Paper I [14], we have derived a general expression for the noise kernel of a quantum scalar field in an arbitrary curved spacetime as products of covariant derivatives of the quantum field's Green function. It is finite when the noise kernel is evaluated for distinct pairs of points (and non-null points for a massless field). We also showed explicitly the trace of the noise kernel vanishes, confirming there is no noise associated with the trace anomaly. This holds regardless of issues of regularization of the noise kernel.

The noise kernel as a two point function of the stress energy tensor diverges as the pair of points are brought together, sharing the ubiquitous ultraviolet divergence present in ordinary (point-defined) quantum field theory. To calculate a regularized noise kernel, similar to what was done before for the simpler stress energy tensor, it is desirable to have at the start an analytic and finite regularized expression for the Green function. When one can carry out a mode decomposition of the invariant operator and find an analytic solution to the mode functions there are established ways to proceed. For such cases the quantum stress tensor and its fluctuations can be determined using some regularization scheme, from the simple normal ordering [15, 16] or smeared field [17] in Minkowski and Casimir spaces to the elegant ζ -function or the powerful point-separation methods, as demonstrated for the Einstein Universe or for the Casimir effect by the authors earlier [17, 18] and others [19, 20, 21]. Unfortunately, there are many important geometries for which an exact analytic expression of the mode functions is not available, such as the Schwarzschild black hole spacetime.

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In this series of papers we work with the covariant point separation regularization method [22, 23, 24] in its modified form [25, 26] to derive a finite expression for the coincident limit of the noise kernel. The expression derived in Paper I [14] for the noise kernel of a scalar field is completely general and can be used with or without consideration of the renormalization of the Green function. Also, the result there for the coincident limit holds for all choices of the Green function and the metric provided that the Green function has a meaningful coincident limit. In this paper and the next one, we apply this formal procedure to specific spacetimes of physical interest. We do this by working with an analytic form of the Green function. When such a form is available one can carry out an end point expansion displaying the ultraviolet divergence. Subtraction of the Hadamard ansatz expressed as a series expansion will render this Green function finite in the coincident limit. With this, one can calculate the noise kernel for a variety of spacetimes.

An analytic form is obtained by invoking the Gaussian approximation introduced by Bekenstein and Parker [27]. For a massless scalar field in an ultra-static spacetime whose metric has an optical form (one where the Euclidean time τ -time component of the metric $g_{\tau\tau} = 1$) this provides a closed expression for the Green function. In this paper we use the Gaussian approximation for the Green function for such quantum fields to evaluate the noise kernel in the following optical metrics : hot flat space, and the optical Schwarzschild spacetime, which is conformal to the Schwarzschild metric. For hot flat space, the Gaussian Green function is exact. For optical Schwarzschild, the Gaussian Green function is known to be a fairly good approximation for calculating the stress tensor [28], which involves second covariant derivatives of the Green function. We will carry out this calculation for the noise kernel which requires up to four covariant derivatives of the Green function. Thus the validity of the Gaussian approximation will be tested to its new limit. A reliable check is provided by the trace of the noise kernel, which for massless conformal fields should be zero.

Thus the goal of this paper is threefold: First, to present the detailed steps in the calculation of the regularized noise kernel for a quantum scalar field in a general curved spacetime using the modified point-separation scheme. Second, to derive the regularized noise kernel for a thermal field in flat space. Third, to determine the range of validity of the Gaussian (Bekenstein-Parker-Page) approximation to the Green function by examining the error in the noise kernel expression for the optical Schwarzschild spacetime.

We present an outline of the calculation as follows: In Section II we give a brief description of the Gaussian approximation to the Green function [27] for ultrastatic spacetimes [28]. In Section III we consider the regularization of the heat kernel in the class of optical metrics. We expand this Green function in an end point series so that it can be separated into a divergent piece and a finite remainder. The divergent term is independent of the approximation since we know this structure must be of a general form given by the Hadamard ansatz. After this is subtracted, we have a series expansion of the renormalized Green function. We then substitute this expansion in the general expression obtained in Paper I for the coincident limit of the noise kernel. The resulting expression is quite long and formal. At this point one can introduce the specific metric of interest and determine the component values of the Green function expansion tensors (by symbolic computation). From this we can readily generate all the needed component values of the coincident limits of the covariant derivatives of the Green function, along with the covariant derivatives of the coincident limits. These explicitly evaluated tensors are then substituted in the general expression obtained in Paper I to get the final result. [36] We give two examples in Section IV. For the case of hot flat space we derive the variance of the energy and pressure density for a quantum field at finite temperature. This is a useful compendium to the results obtained earlier [17] for quantum fields in Minkowski and Casimir geometries in reference to issues like the validity of semiclassical gravity [15, 17, 29, 30]. For a massless, conformally coupled field in the optical Schwarzschild (the ultrastatic spacetime conformal to the Schwarzschild black hole), we obtain for the regularized noise kernel at spatial infinity the same result as a thermal field in flat space, as it should, and a finite result at the horizon in a state conformally-related to the Hartle-Hawking state. However the latter expression computed with the Gaussian approximation has a non-vanishing trace. In Section V we study the nature and source of this error by examining the validity of the Gaussian approximation at successive orders. It works reasonably well to the third covariant derivative order. The inadequacy of the Gaussian approximation to the Green function for the calculation of the noise kernel arises from the Green function's failure to satisfy the field equation of the scalar field at the fourth order.

II. GAUSSIAN APPROXIMATION

We give a brief description of the Gaussian approximation to the Green function for quantum fields in optical spacetimes a la Bekenstein-Parker[27] and Page [28].

In the Schwinger-DeWitt proper-time formalism [31] the Green function is expressed in terms of the heat kernel $K(x, y, s)$ via

$$G(x, y) = \int_0^\infty K(x, y, s) ds. \quad (2.1)$$

where the heat kernel satisfies

$$\left[\frac{\partial}{\partial s} - \left(\square - \frac{R}{6} \right) \right] K(x, y, s) = 0, \quad K(x, y, 0) = \delta(x - y) \quad (2.2)$$

The optical metric for an ultrastatic spacetime has the product form

$$ds^2 = g_{ab} dx^a dx^b = d\tau^2 + g_{ij} dx^i dx^j \quad (2.3)$$

We assume in the Euclidean sector the imaginary time dimension is periodic with period $2\pi/\kappa = 1/T$ with T the temperature. For a black hole, κ is the surface gravity but can be regarded as a temperature parameter here. This form of the metric allows the kernel to take on the product form

$$K(x, y, s) = K_1(\tau, \tau', s) K_3(\mathbf{x}, \mathbf{y}, s) \quad (2.4)$$

with each of the kernels satisfying

$$\left(\frac{\partial}{\partial s} - \frac{\partial^2}{\partial \tau^2} \right) K_1(\tau, \tau', s) = 0, \quad (2.5a)$$

$$\left[\frac{\partial}{\partial s} - \left(\nabla_i \nabla^i - \frac{R}{6} \right) \right] K_3(\mathbf{x}, \mathbf{y}, s) = 0 \quad (2.5b)$$

Equation (2.5a) has the periodic solution

$$K_1(\tau, \tau', s) = \frac{\kappa}{2\pi} \sum_{n=-\infty}^{\infty} \exp(-\kappa^2 n^2 s + i\kappa n \Delta\tau) \quad (2.6)$$

($\Delta\tau = \tau - \tau'$). Equation (2.5b) in general is difficult to solve, but Bekenstein and Parker[27] find an approximate solution using the Gaussian approximation to the path integral representation. For K_3 , it takes the form

$$K_{3\text{Gauss}}(\mathbf{x}, \mathbf{y}, s) = \frac{{}^{(3)}\Delta^{1/2}}{(4\pi s)^{\frac{3}{2}}} \exp\left(-\frac{{}^{(3)}\sigma}{2s}\right) \quad (2.7)$$

where ${}^{(3)}\sigma$ is the world function for the three-dimensional spatial geometry. In the above ${}^{(3)}\Delta^{1/2}$ is the VanVleck-Morette determinant for the spatial geometry. Since we have an optical metric, the four-dimensional world function is $\sigma = {}^{(3)}\sigma + \Delta\tau^2/2$ and there is no difference between the three and four dimensional $\Delta^{1/2}$.

Since the complete Hadamard-Minakshisundaram-Pleijel-DeWitt expansion [22] would be

$$K_3 = K_{3\text{Gauss}} \sum_{n=0}^{\infty} a_n(\mathbf{x}, \mathbf{y}) s^n \quad (2.8)$$

the Gaussian approximation is equivalent to only taking the first term in this power series.

By putting (2.6) and (2.7) back into (2.4) and carrying out the integration (2.1), Page obtains

$$G_{\text{Gauss}}(x, y) = \frac{\kappa \Delta^{1/2}}{8\pi^2 r} \frac{\sinh \kappa r}{(\cosh \kappa r - \cos \kappa \tau)} \quad (2.9)$$

as the Gaussian approximation for the Green function, where $r = (2 {}^{(3)}\sigma)^{\frac{1}{2}}$.

III. NOISE KERNEL IN OPTICAL SPACETIMES

In this section, the noise kernel for an ultrastatic spacetime with an optical metric is determined. For this class of geometries, we start directly with (2.9) for the Green function. The first step is to expand this Green function about the coincident limit. Since the noise kernel has terms with at most four covariant derivatives, this expansion needs to be to fourth order in σ^a , or second order in $\sigma = \sigma^p \sigma_p/2$, and fourth order in $\Delta\tau$. Doing this expansion yields

$$G_{\text{Gauss}} = \frac{\Delta^{\frac{1}{2}}}{8\pi^2 \sigma} + \frac{\Delta^{\frac{1}{2}}}{8\pi^2} \left\{ \frac{\kappa^2}{6} + \frac{\kappa^4}{180} (2\Delta\tau^2 - \sigma) \right\}$$

$$+ \frac{\kappa^6}{3780} (4 \Delta \tau^4 - 6 \Delta \tau^2 \sigma + \sigma^2) \Big\} + O\left(\sigma^{\frac{5}{2}}, \delta \tau^5\right) \quad (3.1)$$

By subtracting from the Gaussian Green function the Hadamard ansatz

$$S(x, y) = \frac{1}{16 \pi^2} \left(\frac{2 \Delta^{\frac{1}{2}}}{\sigma} + \sigma w_1 + \sigma^2 w_2 \right) + O(\sigma^3) \quad (3.2)$$

(the $V(x, x')$ term is absent since there is no $\log \sigma$ divergence present in the expansion of the Gaussian approximation to the Green function) we get the renormalized Green function

$$G_{\text{ren}} = G_{\text{gauss}} - S. \quad (3.3)$$

The divergent term present (3.1) is cancelled by the divergent term from the Hadamard ansatz.

Since our main interest here is to determine the coincident limit of the noise kernel, we next turn to developing the series expansion

$$G_{\text{ren}} = \frac{1}{(4\pi)^2} \left(G_{\text{ren}}^{(0)} + \sigma^{;p} G_{\text{ren}p}^{(1)} + \sigma^{;p} \sigma^{;q} G_{\text{ren}pq}^{(2)} + \sigma^{;p} \sigma^{;q} \sigma^{;r} G_{\text{ren}pqr}^{(3)} + \right. \\ \left. + \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} G_{\text{ren}pqrs}^{(4)} \right) \quad (3.4)$$

of the regularized Green function. With this, it will be straightforward to compute the coincident limits of the various covariant derivatives needed. We start by assuming the expansions

$$\Delta^{\frac{1}{2}} \approx 1 + \sigma^{;p} \sigma^{;q} \Delta_{pq}^{(2)} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \Delta_{pqr}^{(3)} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} \Delta_{pqrs}^{(4)} \quad (3.5a)$$

$$\Delta \tau^2 \approx \sigma^{;p} \sigma^{;q} \delta \tau_{pq}^{(2)} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \delta \tau_{pqr}^{(3)} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} \delta \tau_{pqrs}^{(4)} \quad (3.5b)$$

$$w_1 \approx w_1^{(0)} + \sigma^{;p} w_1^{(1)}{}_p + \sigma^{;p} \sigma^{;q} w_1^{(2)}{}_{pq} \quad (3.5c)$$

$$w_2 \approx w_2^{(0)} \quad (3.5d)$$

The specific values of the expansion tensors in these series are derived in the Appendices. Carrying out the subtraction (3.3) and substituting the expansions (3.5), we find the expansions tensors in (3.4) are

$$G_{\text{ren}}^{(0)} = \frac{\kappa^2}{3} \quad (3.6a)$$

$$G_{\text{ren}a}^{(1)} = 0 \quad (3.6b)$$

$$G_{\text{ren}ab}^{(2)} = \frac{\kappa^2}{3} \Delta_{ab}^{(2)} + \frac{\kappa^4}{180} (4 \delta \tau_{ab}^{(2)} - g_{ab}) - \frac{1}{2} w_1^{(0)} g_{ab} \quad (3.6c)$$

$$G_{\text{ren}abc}^{(3)} = \frac{\kappa^2}{3} \Delta_{abc}^{(3)} + \frac{\kappa^4}{45} \delta \tau_{abc}^{(3)} - \frac{1}{2} g_{ab} w_1^{(1)}{}_c \quad (3.6d)$$

$$G_{\text{ren}abcd}^{(4)} = \frac{\kappa^2}{3} \Delta_{abcd}^{(4)} + \frac{\kappa^4}{180} (4 \Delta_{ab}^{(2)} \delta \tau_{cd}^{(2)} + 4 \delta \tau_{abcd}^{(4)} - \Delta_{cd}^{(2)} g_{ab}) \\ + \frac{\kappa^6}{7560} (16 \delta \tau_{ab}^{(2)} \delta \tau_{cd}^{(2)} - 12 \delta \tau_{cd}^{(2)} g_{ab} + g_{ab} g_{cd}) \\ - \frac{1}{4} g_{ab} (w_2^{(0)} g_{cd} + 2 w_1^{(2)}{}_{cd}) \quad (3.6e)$$

Using the explicit forms of the expansion tensor values, (C8), (C10), (C12a) and (D5), we get

$$G_{\text{ren}ab}^{(2)} = \frac{\kappa^2}{36} R_{ab} + \frac{\kappa^4}{180} (4 \delta_a{}^\tau \delta_b{}^\tau - g_{ab}) \quad (3.7a)$$

$$G_{\text{ren}abc}^{(3)} = -\frac{\kappa^2}{72} R_{ab;c} + \frac{\kappa^4}{45} \Gamma_{ab}{}^\tau \delta_c{}^\tau \quad (3.7b)$$

$$G_{\text{ren}abcd}^{(4)} = \frac{\kappa^2}{4320} (18 R_{ab;cd} + 5 R_{ab} R_{cd} + 4 R_{paqb} R_c{}^p{}_d{}^q) \\ + \frac{\kappa^4}{2160} (12 \Gamma_{ab}{}^\tau \Gamma_{cd}{}^\tau - 16 \Gamma_{ab;c}{}^\tau \delta_d{}^\tau + 4 \delta_a{}^\tau \delta_b{}^\tau R_{cd} - g_{ab} R_{cd})$$

$$\begin{aligned}
& + \frac{\kappa^6}{7560} (16 \delta_a^\tau \delta_b^\tau \delta_c^\tau \delta_d^\tau - 12 \delta_a^\tau \delta_b^\tau g_{cd} + g_{ab} g_{cd}) \\
& - \frac{1}{4} g_{ab} \left(w_2^{(0)} g_{cd} + 2 w_1^{(2)}{}_{cd} \right)
\end{aligned} \tag{3.7c}$$

where we have also used

$$w_1^{(0)} = -\frac{1}{240} (R_{;p}{}^p - R_{pq} R^{pq} + R_{pqrs} R^{pqrs}) = 0, \tag{3.8}$$

along with $w_1^{(1)}{}_a = 0$, which hold for ultrastatic metrics.

For flat space, these tensors reduce to

$$G_{\text{ren}ab}^{(2)} = \frac{\kappa^4}{180} (-\eta_{ab} + 4 \eta_a^\tau \eta_b^\tau) \tag{3.9a}$$

$$G_{\text{ren}abc}^{(3)} = 0 \tag{3.9b}$$

$$G_{\text{ren}abcd}^{(4)} = \frac{\kappa^6}{7560} (\eta_{ab} \eta_{cd} - 12 \eta_a^\tau \eta_b^\tau \eta_{cd} + 16 \eta_a^\tau \eta_b^\tau \eta_c^\tau \eta_d^\tau) \tag{3.9c}$$

Now that we know the end point series expansion (3.4) of G_{ren} , the coincident limit of terms with up to four covariant derivatives are computed. We simply differentiate the series (3.4) and then use the results from Appendix (A) for the coincident limits of the covariant derivatives of the world function σ . The results are

$$16 \pi^2 [G_{\text{ren}}] = G_{\text{ren}}^{(0)} \tag{3.10a}$$

$$16 \pi^2 [G_{\text{ren};a}] = G_{\text{ren};a}^{(0)} \tag{3.10b}$$

$$16 \pi^2 [G_{\text{ren};ab}] = G_{\text{ren};(ab)}^{(0)} + 2 G_{\text{ren}(ab)}^{(2)} \tag{3.10c}$$

$$16 \pi^2 [G_{\text{ren};abc}] = 6 \left(G_{\text{ren}(ab;c)}^{(2)} + G_{\text{ren}(abc)}^{(3)} \right) + G_{\text{ren};abc}^{(0)} \tag{3.10d}$$

$$\begin{aligned}
16 \pi^2 [G_{\text{ren};abcd}] &= 12 \left(2 G_{\text{ren}(abc;d)}^{(3)} + G_{\text{ren}(ab;cd)}^{(2)} + 2 G_{\text{ren}(abcd)}^{(4)} \right) \\
&+ \frac{2}{3} \left(G_{\text{ren}pa}^{(2)} (R_{bcd}{}^p - 2 R_{bdc}{}^p) + G_{\text{ren}pb}^{(2)} (R_{acd}{}^p - 2 R_{adc}{}^p) \right. \\
&\quad \left. + G_{\text{ren}pc}^{(2)} (R_{abd}{}^p - 2 R_{adb}{}^p) + G_{\text{ren}pd}^{(2)} (R_{abc}{}^p - 2 R_{acb}{}^p) \right) \\
&+ G_{\text{ren};abcd}^{(0)}
\end{aligned} \tag{3.10e}$$

We now have all the information we need to compute the coincident limit of the noise kernel (see Eqn (3.24) of Paper I). Since the point separated noise kernel $N_{abc'd'}(x, y)$ involves covariant derivatives at the two points at which it has support, when we take the coincident limit we can use Synge's theorem to move the derivatives acting at the second point y to ones acting at the first point x . Due to the length of the expression for the noise kernel, we will here give an example of the calculation by examining a single term. The complete expression for the coincident limit of the point separated noise kernel can be found in Paper I as Eqn (4.16).

Consider a typical term from the noise kernel functional (Eqn (3.24) of Paper I):

$$G_{\text{ren};c'b} G_{\text{ren};d'a} + G_{\text{ren};c'a} G_{\text{ren};d'b} \tag{3.11}$$

As was derived in Paper I, the noise kernel itself is related to the noise kernel functional via

$$N_{abc'd'} = N_{abc'd'} [G_{\text{ren}}(x, y)] + N_{abc'd'} [G_{\text{ren}}(y, x)]. \tag{3.12}$$

We account for this by adding the same term but with the roles of x and y reversed. Taken together, we need to analyze

$$G_{\text{ren};c'b} G_{\text{ren};d'a} + G_{\text{ren};a'd} G_{\text{ren};b'c} + G_{\text{ren};c'a} G_{\text{ren};d'b} + G_{\text{ren};a'c} G_{\text{ren};b'd}, \tag{3.13}$$

in particular, its coincident limit:

$$[G_{\text{ren};c'b}] [G_{\text{ren};d'a}] + [G_{\text{ren};a'd}] [G_{\text{ren};b'c}] + [G_{\text{ren};c'a}] [G_{\text{ren};d'b}] + [G_{\text{ren};a'c}] [G_{\text{ren};b'd}] \tag{3.14}$$

We apply Synge's theorem to remove any explicit reference to the point y ,

$$([G_{\text{ren};a}]_{;d} - [G_{\text{ren};ad}]) ([G_{\text{ren};b}]_{;c} - [G_{\text{ren};bc}])$$

$$\begin{aligned}
& + ([G_{\text{ren}};d]_a - [G_{\text{ren}};ad]) ([G_{\text{ren}};c]_b - [G_{\text{ren}};bc]) \\
& + ([G_{\text{ren}};a]_c - [G_{\text{ren}};ac]) ([G_{\text{ren}};b]_d - [G_{\text{ren}};bd]) \\
& + ([G_{\text{ren}};c]_a - [G_{\text{ren}};ac]) ([G_{\text{ren}};d]_b - [G_{\text{ren}};bd])
\end{aligned} \tag{3.15}$$

and express the results in terms of the expansion tensors (3.10e) and find

$$8 \left(G_{\text{ren}ad}^{(2)} G_{\text{ren}bc}^{(2)} + G_{\text{ren}ac}^{(2)} G_{\text{ren}bd}^{(2)} \right) \tag{3.16}$$

Though this may look relatively compact, when we use the results (3.7c) to get the final form, in terms of the local geometry, this is no longer the case. Making these substitutions, our pair of terms become

$$\begin{aligned}
& \frac{\kappa^4}{2592 \pi^2} (R_{ad} R_{bc} + R_{ac} R_{bd}) \\
& + \frac{\kappa^6}{12960 \pi^2} \{ (4 \delta_b^\tau \delta_d^\tau - g_{bd}) R_{ac} + (4 \delta_b^\tau \delta_c^\tau - g_{bc}) R_{ad} \\
& \quad + (4 \delta_a^\tau \delta_d^\tau - g_{ad}) R_{bc} + (4 \delta_a^\tau \delta_c^\tau - g_{ac}) R_{bd} \} \\
& + \frac{\kappa^8}{64800 \pi^2} \{ 32 \delta_a^\tau \delta_b^\tau \delta_c^\tau \delta_d^\tau + g_{ad} g_{bc} + g_{ac} g_{bd} \\
& \quad - 4 (\delta_b^\tau \delta_d^\tau g_{ac} + \delta_b^\tau \delta_c^\tau g_{ad} + \delta_a^\tau \delta_d^\tau g_{bc} + \delta_a^\tau \delta_c^\tau g_{bd}) \}
\end{aligned} \tag{3.17}$$

The noise kernel functional consists of 25 such terms, some quite a bit more involved. This is especially true when a single Green function has four derivatives acting on it. To gain insight into the physics we work with some specific spacetime.[37]

When we do choose a metric, we proceed by directly evaluating the components of the expansion tensors (3.7c). Once these expansion tensor components are known, the actual components of the coincident limits of the covariant derivatives of G_{ren} are in turn computed using (3.10e). Then it is straightforward from there to get the covariant derivatives of the coincident limits, since application of Synge's theorem will move derivatives acting at y such that we also need the covariant derivatives of the coincident limits. Now that we have the component values of all the covariant derivatives of the various coincident limits of the regularized Green function G_{ren} , we substitute them in the coincident limit expression for the noise kernel, Eqn (4.16) of Paper I, and arrive at the final result we seek.

Before turning to specific metrics and as a check, we can reproduce the derivation of the renormalized stress tensor. We start with the point separated expression for the stress tensor, which for a massless, conformally coupled scalar field is

$$\begin{aligned}
\langle T_{ab}(x, y) \rangle &= \frac{1}{3} \left(g^{p'b} G_{\text{ren};p'a} + g^{p'a} G_{\text{ren};p'b} \right) - \frac{1}{6} g^{p'q} G_{\text{ren};p'q} g_{ab} \\
& - \frac{1}{6} \left(g^{p'a} g^{q'b} G_{\text{ren};p'q'} + G_{\text{ren};ab} \right) + \frac{1}{6} \left((G_{\text{ren};p'}^{p'} + G_{\text{ren};p}^p) g_{ab} \right) \\
& + \frac{1}{6} G_{\text{ren}} \left(\frac{-(R g_{ab})}{2} + R_{ab} \right)
\end{aligned} \tag{3.18}$$

We take the coincident limit and utilize Synge's theorem to obtain

$$\begin{aligned}
\langle T_{ab} \rangle_{\text{ren}} &= \frac{1}{6} (3 [G_{\text{ren}};a]_b + 3 [G_{\text{ren}};b]_a - [G_{\text{ren}}]_{ab} - 6 [G_{\text{ren}};ab]) \\
& - \frac{1}{6} (3 [G_{\text{ren}};p]^{;p} - [G_{\text{ren}}]_{;p}^p - 3 [G_{\text{ren};p}^p]) g_{ab} \\
& - \frac{1}{12} [G_{\text{ren}}] (R g_{ab} - 2 R_{ab}) \\
& = -\frac{1}{6} \left(G_{\text{ren};ab}^{(0)} + 12 G_{\text{ren}ab}^{(2)} - G_{\text{ren}}^{(0)} R_{ab} \right) \\
& - \frac{1}{12} \left(G_{\text{ren}}^{(0)} R - 2 G_{\text{ren};p}^{(0)p} - 12 G_{\text{ren}p}^{(2)p} \right) g_{ab}
\end{aligned} \tag{3.19}$$

With the explicit values (3.7c) for the expansion tensors we recover the familiar result:

$$\langle T_{ab} \rangle_{\text{ren}} = \frac{\kappa^4}{1440 \pi^2} (g_{ab} - 4 \delta_a^\tau \delta_b^\tau) \tag{3.20}$$

IV. EXAMPLES

A. Hot Flat Space

The first example we consider is that of a finite temperature $T = \kappa/2\pi$ quantum scalar field in flat space. With this, the stress tensor takes the usual form

$$\langle T_{ab} \rangle = \text{diag}\{-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1\}\rho, \quad \rho = -\frac{\pi^2 T^4}{30} \quad (4.1)$$

with $x^a = (x, y, z, \tau)$.

Using (3.9c) and (3.10e), the non-zero coincident limits of the derivatives of G_{ren} are

$$[G_{\text{ren}}] = \frac{T^2}{12} \quad (4.2a)$$

$$[G_{\text{ren};ab}] = -\frac{(\pi^2 T^4)}{90} (\delta_{ab} - 4\delta_a^\tau \delta_b^\tau) \quad (4.2b)$$

$$[G_{\text{ren};abcd}] = \frac{4\pi^4 T^6}{315} (\delta_{ab}\delta_{cd} - 12\delta_a^\tau \delta_b^\tau \delta_{cd} + 16\delta_a^\tau \delta_b^\tau \delta_c^\tau \delta_d^\tau) \quad (4.2c)$$

Taking the coincident limit of the massless case for the noise kernel, Eqn (4.16) of Paper I, (for this case, we keep ξ arbitrary), we find

$$\begin{aligned} N_{abcd} = & \frac{\pi^4 T^8}{226800} \{ 32 (7 - 28\xi + 162\xi^2) g_a^\tau g_b^\tau g_c^\tau g_d^\tau \\ & - 8 (7 - 42\xi + 130\xi^2) (g_{cd} g_a^\tau g_b^\tau + g_{ab} g_c^\tau g_d^\tau) \\ & - 4 (7 - 28\xi + 148\xi^2) (g_{bd} g_a^\tau g_c^\tau + g_{ad} g_b^\tau g_c^\tau + g_{bc} g_a^\tau g_d^\tau + g_{ac} g_b^\tau g_d^\tau) \\ & + 4 (7 - 63\xi + 167\xi^2) g_{ab} g_{cd} + (7 - 28\xi + 108\xi^2) (g_{ad} g_{bc} + g_{ac} g_{bd}) \} \end{aligned} \quad (4.3)$$

From this, we can immediately compute the trace:

$$N_p{}^p{}_q{}^q = \frac{\pi^4 T^8}{1350} (-1 + 6\xi)^2 \quad (4.4)$$

which we see vanishes for the conformal coupling case ($\xi = 1/6$). The non-vanishing components for general coupling are

$$\begin{aligned} N_{\tau\tau\tau\tau} &= \frac{\pi^4 T^8}{37800} (7 - 14\xi + 270\xi^2) \xrightarrow{\xi \rightarrow 0} \frac{\pi^4 T^8}{5400} = \frac{1}{6} \langle T_{\tau\tau} \rangle^2 \\ N_{xxxx} &= \frac{\pi^4 T^8}{113400} (21 - 154\xi + 442\xi^2) \xrightarrow{\xi \rightarrow 0} \frac{\pi^4 T^8}{5400} = \frac{3}{2} \langle T_{xx} \rangle^2 \\ N_{\tau\tau xx} &= -\frac{\pi^4 T^8}{56700} (7 - 21\xi + 93\xi^2) \xrightarrow{\xi \rightarrow 0} -\frac{\pi^4 T^8}{8100} = \frac{1}{3} \langle T_{\tau\tau} \rangle \langle T_{xx} \rangle \\ N_{xxyy} &= \frac{\pi^4 T^8}{56700} (7 - 63\xi + 167\xi^2) \xrightarrow{\xi \rightarrow 0} \frac{\pi^4 T^8}{8100} = 1 \langle T_{xx} \rangle^2 \\ N_{\tau x \tau x} &= -\frac{\pi^4 T^8}{226800} (21 - 84\xi + 484\xi^2) \xrightarrow{\xi \rightarrow 0} -\frac{\pi^4 T^8}{10800} = \frac{1}{4} \langle T_{\tau\tau} \rangle \langle T_{xx} \rangle \\ N_{xyxy} &= \frac{\pi^4 T^8}{226800} (7 - 28\xi + 108\xi^2) \xrightarrow{\xi \rightarrow 0} \frac{\pi^4 T^8}{32400} = \frac{1}{4} \langle T_{xx} \rangle^2 \end{aligned} \quad (4.5)$$

and those that follow from the symmetry of the metric.

We use the dimensionless measure of fluctuations [15, 16, 17, 18] (this is not a tensor, but a measure of the fluctuations for each component):

$$\Delta_{abcd} = \left| \frac{\langle T_{ab} T_{cd} \rangle - \langle T_{ab} \rangle \langle T_{cd} \rangle}{\langle T_{ab} T_{cd} \rangle} \right| = \left| \frac{4N_{abcd}}{4N_{abcd} + \langle T_{ab} \rangle \langle T_{cd} \rangle} \right| \quad (4.6)$$

From inspection, $0 \leq \Delta_{abcd} \leq 1$. Only for $\Delta \sim 0$ can the fluctuations be viewed as small. For $\Delta \sim 1$ the fluctuations are comparable to the mean value.

For hot flat space, the results for Δ are

| $abcd$ | Δ_{abcd} | $\xi = 0$ | $\xi = \frac{1}{6}$ |
|--------------------|--|-------------------------|-----------------------------|
| $\tau\tau\tau\tau$ | $\frac{2(7-14\xi+270\xi^2)}{35-28\xi+540\xi^2}$ | $\frac{2}{5} \sim 0.4$ | $\frac{73}{136} \sim 0.54$ |
| $xxxx$ | $\frac{2(21-154\xi+442\xi^2)}{49-308\xi+884\xi^2}$ | $\frac{6}{7} \sim 0.86$ | $\frac{137}{200} \sim 0.69$ |
| $\tau\tau xx$ | $\frac{4(7-21\xi+93\xi^2)}{49-84\xi+372\xi^2}$ | $\frac{4}{7} \sim 0.57$ | $\frac{73}{136} \sim 0.54$ |
| $xxyy$ | $\frac{4(7-63\xi+167\xi^2)}{35-252\xi+668\xi^2}$ | $\frac{4}{5} \sim 0.8$ | $\frac{41}{104} \sim 0.39$ |

(4.7)

From these results we see, even for the simple case of thermal fluctuations in flat space, the fluctuations present in the stress tensor are important. Discussions on the implication of the fluctuation to mean ratio can be found in [15, 16, 17, 29, 30].

B. Optical Schwarzschild black hole

We now consider the optical spacetime conformally related to the Schwarzschild black hole spacetime. For this spacetime, the line element is

$$ds^2 = d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 + \left(1 - \frac{2M}{r}\right)^{-1} r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (4.8)$$

Taking $\kappa = 2\pi T$ and $T = 1/(8\pi M)$ we choose the quantum state corresponding to the Hartle-Hawking state in the conformally-related Schwarzschild spacetime. We use the spacetime coordinates $x^a = (r, \theta, \phi, \tau)$ and introduce the rescaled inverse radial coordinate $x = 2M/r = 1/(4\pi T r)$. Spatial infinity corresponds to $x = 0$ and $x = 1$ is the black hole horizon. For a massless conformal scalar field ($m = 0$, $\xi = 1/6$) the stress tensor is

$$\langle T_a{}^b \rangle = \text{diag} \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1 \right\} \frac{\pi^2 T^4}{30} \quad (4.9)$$

We recover the standard thermal result for the stress tensor. The component values for the noise kernel are

$$N_{\tau}{}^{\tau}{}_{\tau}{}^{\tau} = \frac{\pi^4 T^8}{2041200} (657 - 1050 x^4 + 8400 x^6 - 16800 x^7 - 16997400 x^8 + 80206000 x^9 - 140910000 x^{10} + 109242000 x^{11} - 31515750 x^{12}) \quad (4.10a)$$

$$N_r{}^r{}_{r}{}^r = \frac{\pi^4 T^8}{2041200} (137 + 1120 x^3 - 210 x^4 + 80640 x^5 - 103600 x^6 + 16800 x^7 + 9829800 x^8 - 43285200 x^9 + 73602000 x^{10} - 57186000 x^{11} + 17057250 x^{12}) \quad (4.10b)$$

$$N_{\theta}{}^{\theta}{}_{\theta}{}^{\theta} = \frac{\pi^4 T^8}{2041200} (137 - 560 x^3 + 1470 x^4 + 30240 x^5 - 44800 x^6 - 16800 x^7 - 15805800 x^8 + 73203600 x^9 - 127386000 x^{10} + 98820000 x^{11} - 28788750 x^{12}) \quad (4.10c)$$

$$N_{\tau}{}^{\tau}{}_{r}{}^r = \frac{\pi^4 T^8}{6123600} (-657 + 6720 x^3 - 5670 x^4 - 42000 x^6 + 484920 x^8 + 4202480 x^9 - 12514800 x^{10} + 9439200 x^{11} - 1548450 x^{12}) \quad (4.10d)$$

$$N_{\tau}{}^{\tau}{}_{\theta}{}^{\theta} = \frac{\pi^4 T^8}{6123600} (-657 - 3360 x^3 + 4410 x^4 + 8400 x^6 + 25200 x^7 - 53478360 x^8 + 248828960 x^9 - 434589600 x^{10} + 337051800 x^{11} - 97825050 x^{12}) \quad (4.10e)$$

$$N_r{}^r{}_{\theta}{}^{\theta} = \frac{\pi^4 T^8}{6123600} (123 - 5040 x^3 + 3150 x^4 - 120960 x^5 + 176400 x^6 - 25200 x^7 + 2858760 x^8 - 10502560 x^9 + 15885600 x^{10} - 12447000 x^{11} + 4178250 x^{12}) \quad (4.10f)$$

From the component values of the noise kernel we can compute its trace

$$N = N_p{}^p{}_q{}^q = -\frac{4\pi^4 T^8 x^8}{567} (9720 - 45832 x + 80520 x^2 - 62424 x^3 + 18009 x^4) \quad (4.11)$$

We know from prior results that this should vanish, since we have worked with a massless conformally coupled scalar field. This failure of the trace of the noise kernel to vanish is due to the failure of the Gaussian approximated Green function (2.9) to satisfy the field equation to fourth order.

V. FAILURE OF GAUSSIAN APPROXIMATION AT THE FOURTH ORDER

To be sure that this error does not arise from the symbolic manipulation, let us mention ways to check the correctness of the algorithm. The basic procedures for generating the needed series expansions are recursive on the expansion

order. (The recursion formulas of the expansion used in point-separation are collected in the Appendices). For the noise kernel we need results up to fourth order in the separation distance. The well established work for the stress tensor is to second order. This provides a check of our code by verifying we always get the known results for the stress tensor expectation value. Once we know the second order recursion is correct, we know the algorithm is functioning as desired. The correctness in the (new) fourth order terms becomes particularly important in the work in Paper III, when we consider metrics conformally related, as we get intermediate results of up to 1100 terms in length.

We can check the accuracy of the Gaussian approximation by using the computed component values of coincident limit of the covariant derivatives of G_{ren} . We have assumed

$$G_{\text{ren};p}{}^p - \frac{G_{\text{ren}} R}{6} = 0 \quad (5.1)$$

This can be used to test the approximation order by order. To test up to the second order, we just take the coincident limit

$$[G_{\text{ren};p}{}^p] - \frac{R [G_{\text{ren}}]}{6} = 0. \quad (5.2)$$

For the metric (4.8), the scalar curvature is

$$R = -\frac{6 M^2}{r^4} = -24 \pi^2 T^2 x^4 \quad (5.3)$$

while the results of our computation of the component values of the coincident limit of the covariant derivatives of G_{ren} yield

$$[G_{\text{ren}}] = \frac{4 \pi^2 T^2}{3}, \quad \text{and} \quad [G_{\text{ren};p}{}^p] = -\frac{16 \pi^4 T^4 x^4}{3}. \quad (5.4)$$

With these values, Eq. (5.2) can be seen to be satisfied. Thus the Gaussian approximation is good to the second order. (This had better be the case, as the approximation at this order has been checked against numerical computations of the stress tensor by other authors. See e.g., description in [32, 33])

To check the third order term, we take one covariant derivative and then the coincident limit of (5.1):

$$[G_{\text{ren};p}{}^p{}_a] - \frac{1}{6} (R_{;a} [G_{\text{ren}}] + R [G_{\text{ren};a}]) = 0 \quad (5.5)$$

Using the results $[G_{\text{ren};a}] = 0$,

$$R_{;a} = \{384 \pi^3 T^3 x^5, 0, 0, 0\} \quad (5.6)$$

and

$$[G_{\text{ren};p}{}^p{}_a] = \left\{ \frac{256 \pi^5 T^5 x^5}{3}, 0, 0, 0 \right\} \quad (5.7)$$

we see Eq. (5.5) is also satisfied. This has to be the case: (3.7c) shows the third order expansion tensor does not have any contribution from the $W(x, y)$ part of the Hadamard ansatz and this is the only place a lack of symmetry in $G_{\text{ren}}(x, y)$ could come in. Therefore the Gaussian Green function (2.9) is symmetric. For symmetric functions, the odd order expansion tensors are determined completely by the even order tensors (see (B16a)).

Continuing to fourth order, (5.1) becomes

$$\begin{aligned} [G_{\text{ren};p}{}^p{}_a{}^b] - \frac{1}{6} (R_{;a}{}^b [G_{\text{ren}}] + R^{;b} [G_{\text{ren};a}] + R_{;a} [G_{\text{ren}}{}^{;b}] + R [G_{\text{ren};a}{}^b]) = \\ [G_{\text{ren};p}{}^p{}_a{}^b] - \frac{1}{6} (R_{;a}{}^b [G_{\text{ren}}] + R [G_{\text{ren};a}{}^b]) = 0 \end{aligned} \quad (5.8)$$

Proceeding as before and evaluating the left hand side above, the component values are

$$\text{diag} \left\{ -\frac{128 \pi^6 T^6 x^8}{315} (162648 - 746888 x + 1295880 x^2 - 1005480 x^3 + 293805 x^4), \right. \\ \left. \frac{128 \pi^6 T^6 x^8}{315} (8424 - 29704 x + 44040 x^2 - 34560 x^3 + 11835 x^4), \right.$$

$$\frac{128\pi^6 T^6 x^8}{315} (8424 - 29704x + 44040x^2 - 34560x^3 + 11835x^4),$$

$$\frac{128\pi^6 T^6 x^8}{189} (9720 - 45832x + 80520x^2 - 62424x^3 + 18009x^4) \} \quad (5.9)$$

The failure of the left hand side of (5.8) to vanish shows that the failure of the trace to vanish comes from the limitations of the Gaussian approximation. The Gaussian approximation is only useful up to the third order in σ^a .

With this knowledge, the trace (4.11) becomes our measure of the error in the noise kernel from the use of the Gaussian approximation. It is important to note that the noise kernel trace N vanishes as $x \rightarrow 0$, or, $r \rightarrow \infty$, i.e. where one would expect the effects of curvature to vanish. We can also see from (4.10f) that the noise is finite at the horizon ($x = 1$).

Using our derived expression for the noise kernel we see that its trace vanishes at spatial infinity, thus we can trust our results there. Using the measure (4.6), the fluctuations at $r \rightarrow \infty$ are

$$\begin{array}{l} abcd : \quad \tau\tau\tau\tau \quad rrrr \quad \theta\theta\theta\theta \quad \tau\tau rr \quad \tau\tau\theta\theta \quad rr\theta\theta \\ \Delta_{abcd} : \quad \frac{73}{136} \quad \frac{137}{200} \quad \frac{137}{200} \quad \frac{73}{136} \quad \frac{73}{136} \quad \frac{41}{104} \end{array} \quad (5.10)$$

which match exactly the results (4.7) for hot flat space with conformal coupling, another reassuring fact. Since the computation of the noise kernel for metric (4.8) is much more involved, this provides yet another check of our symbolic computer code.

Now having shown that it is truly the Gaussian approximation that is at fault for the failure of the noise kernel trace to vanish, we can use its value as a measure of the error in the results (4.10f). Since N should be zero, N/N_{abcd} is a dimensionless measure of this error. At the horizon ($x = 1$),

$$\begin{array}{l} abcd : \quad \tau\tau\tau\tau \quad rrrr \quad \theta\theta\theta\theta \quad \tau\tau rr \quad \tau\tau\theta\theta \quad rr\theta\theta \\ \frac{N_{p^p q^q}}{N_{a^a b^b c^c d^d}} : \quad 627\% \quad 791\% \quad 791\% \quad 1390\% \quad 1390\% \quad 19855\% \end{array} \quad (5.11)$$

These errors show the Gaussian approximation fails to provide reliable results for the noise kernel near the horizon of the optical Schwarzschild metric. We expect this be the case also for the Schwarzschild metric near the horizon, an explicit calculation will appear in the next paper. In the above we have identified the occurrence of significant error begins at the fourth covariant derivative order.

In conclusion, towards the three goals set for this paper, we have detailed the steps in implementing the modified point separation scheme under the Gaussian approximation for the Green function for a quantum scalar field in the optical spacetimes. We have derived the regularized noise kernel for a thermal field in flat space. We have obtained a finite expression for the noise kernel at the horizon of an optical Schwarzschild spacetime and recovered the hot flat space result at infinity. From the error in the noise kernel at the horizon we showed that the Gaussian approximation scheme of Bekenstein-Parker-Page applied to the Green function which provides surprisingly good results for the stress tensor involving the second covariant derivative order of the Green function, fails at the fourth covariant derivative order.

Acknowledgments

NGP thanks Professor R. M. Wald for correspondence on his regularization procedures, and Professors S. Christensen and L. Parker for the use of the MathTensor program. BLH thanks Professors Paul Anderson, Sukanya Sinha and E. Verdaguer for discussions. This work is supported in part by NSF grant PHY98-00967.

APPENDIX A: WORLD FUNCTION σ

In this appendix, we review the properties of the world function σ . We also demonstrate how symbolic computations are implemented and used in this work. Christensen's [24] method for determining the coincident limit of covariant derivatives of functions defined via a covariant differential equation is reviewed.

The world function is defined to be one half of the square of the geodesic distance between two points on a differential manifold. As such, it satisfies the equation

$$\sigma - \frac{\sigma_{;p} \sigma^{;p}}{2} = 0 \quad (\text{A1})$$

along with the initial value

$$[\sigma_{;a}] = 0 \quad (\text{A2})$$

To determine $[\sigma_{;ab}]$, we take two derivatives of (A1). To help illustrate how these calculations are done on the computer, the output presented here is direct output from MathTensor[34], with Mathematica[35] carrying out the formatting. We use the MathTensor function `CD` and get

$$\sigma_{;ab} - \frac{\sigma_{;pb} \sigma_{;a}^p}{2} - \frac{\sigma_{;pa} \sigma_{;b}^p}{2} - \frac{\sigma^{;p} \sigma_{;pab}}{2} - \frac{\sigma_{;p} \sigma_{;a}^p \sigma_{;b}}{2} = 0 \quad (\text{A3})$$

This is put into a canonical form via `Canonicalize`

$$\sigma_{;ab} - \sigma_{;pa} \sigma_{;b}^p - \sigma_{;p} \sigma_{;a}^p \sigma_{;b} = 0 \quad (\text{A4})$$

The condition (A2) is encoded into Mathematica by defining a rule

`Ci[CD[sigma, la_]] -> 0`, where `Ci[]` is a function defined to represent the coincident limit and is formatted to be displayed using the standard $[\dots]$ notation. Using this rule, the coincident limit is

$$[\sigma_{;ab}] - [\sigma_{;pa}] [\sigma_{;b}^p] = 0 \quad (\text{A5})$$

This immediately shows $[\sigma_{;ab}] = g_{ab}$ and such a rule is defined.

Proceeding with one more covariant derivative,

$$0 = \sigma_{;abc} - \sigma_{;pc} \sigma_{;a}^p \sigma_{;b} - \sigma_{;pb} \sigma_{;a}^p \sigma_{;c} - \sigma_{;pa} \sigma_{;b}^p \sigma_{;c} - \sigma_{;p} \sigma_{;a}^p \sigma_{;b} \sigma_{;c}, \quad (\text{A6})$$

and using the two rules we already have, we recursively get

$$0 = [\sigma_{;abc}] + [\sigma_{;acb}] \quad (\text{A7})$$

Using MathTensors' `OrderCD`, which commutes the covariant derivatives on each term until they are in alphabetical order, we get the result

$$\begin{aligned} 0 &= 2 [\sigma_{;abc}] + [\sigma_{;p}] R_a^p{}_{bc} \\ &= 2 [\sigma_{;abc}] \end{aligned} \quad (\text{A8})$$

where (A2) is used to go from the first to the second line. We have the coincident limit of three covariant derivatives acting on σ vanishes.

Proceeding with one more covariant derivative:

$$\begin{aligned} 0 &= \sigma_{;pad} \sigma_{;b}^p \sigma_{;c} + \sigma_{;pac} \sigma_{;b}^p \sigma_{;d} + \sigma_{;pab} \sigma_{;c}^p \sigma_{;d} - \sigma_{;abcd} + \sigma_{;pd} \sigma_{;a}^p \sigma_{;b} \sigma_{;c} \\ &\quad + \sigma_{;pc} \sigma_{;a}^p \sigma_{;b} \sigma_{;d} + \sigma_{;pb} \sigma_{;a}^p \sigma_{;c} \sigma_{;d} + \sigma_{;pa} \sigma_{;b}^p \sigma_{;c} \sigma_{;d} + \sigma_{;p} \sigma_{;a}^p \sigma_{;b} \sigma_{;c} \sigma_{;d} \\ &\Rightarrow \\ 0 &= [\sigma_{;abcd}] + [\sigma_{;acbd}] + [\sigma_{;adbc}] \end{aligned} \quad (\text{A9})$$

where once again we use the rules we already know. Now commuting the covariant derivatives

$$\begin{aligned} 0 &= 3 [\sigma_{;abcd}] + (R_a^p{}_{bc;d} + R_a^p{}_{bd;c}) [\sigma_{;p}] \\ &\quad + [\sigma_{;pd}] R_a^p{}_{bc} + [\sigma_{;pc}] R_a^p{}_{bd} + [\sigma_{;pb}] R_a^p{}_{cd} + [\sigma_{;pa}] R_b^p{}_{cd} \end{aligned} \quad (\text{A10})$$

and using the known rules, we get the equation

$$0 = 3 [\sigma_{;abcd}] + R_{acbd} + R_{adbc} \quad (\text{A11})$$

which we can solve for the term we need. In practice, Mathematica's function `Solve[]` is used, giving us the rule

$$[\sigma; abcd] \rightarrow -\frac{1}{3} (R_{acbd} + R_{adbc}) \quad (\text{A12})$$

By knowing the coincident limit of $n - 1$ covariant derivatives of σ , we determine the coincident limit of n covariant derivatives. This is the recursive algorithm developed by Christensen. It is the main idea we use for computing the expansions needed in this work. A general outline starts by assuming we have the rules for $n - 1$ covariant derivatives, then,

1. Take n covariant derivatives of the defining equation (in this case, Eq (A1));
2. Use the rules for $n - 1$ covariant derivatives to get the coincident limit;
3. Commute the covariant derivatives;
4. Use again the rules for $n - 1$ covariant derivatives on the terms generated;
5. Solve for the coincident limit of the n derivative term;
6. Define a new rule for this term.

These steps are iterated until all terms needed are generated.

For the world function σ , we need to carry this out to eight covariant derivatives. The seventh and eighth order (derivative) terms become quite large. In fact, when computing these expression, we only substitute (steps 2 and 4 above) up to four covariant derivatives and still get results with 240 terms for the seven derivative result and 1101 for the eight. We only finish carrying out the recursion when we use these highest order values.

The results for five and six covariant derivatives are:

$$[\sigma; abcde] = -\frac{1}{4} (R_{acbd;e} + R_{acbe;d} + R_{adbc;e} + R_{adbce;c} + R_{aebc;d} + R_{aebd;c}) \quad (\text{A13})$$

$$\begin{aligned} [\sigma; abcdef] = & - (R_{acbd;ef} + R_{acbe;df} + R_{acbf;de} + R_{adbc;ef} + R_{adbce;cf} \\ & + R_{adbf;ce} + R_{aebc;df} + R_{aebd;cf} + R_{aebf;cd} + R_{afbc;de} \\ & + R_{afbd;ce} + R_{afbe;cd}) / 5 \\ & - (R_{pecd} R_{abf}^p + R_{pebd} R_{acf}^p + R_{pebc} R_{adf}^p + R_{pead} R_{bcf}^p \\ & + R_{peac} R_{bdf}^p + R_{peab} R_{cdf}^p) / 9 \\ & + (R_{pdce} R_{abe}^p + R_{pdce} R_{abf}^p + R_{pdbf} R_{ace}^p + R_{pdbe} R_{acf}^p \\ & + R_{pdaf} R_{bce}^p + R_{pdae} R_{bcf}^p) / 45 \\ & + (R_{pcbf} (R_{ade}^p + 7 R_{aed}^p) + R_{pcbe} (R_{adf}^p + 7 R_{afd}^p) \\ & + R_{pcbd} (R_{aef}^p + 7 R_{afe}^p) + R_{pcaf} (R_{bde}^p + 7 R_{bed}^p) \\ & + R_{pcae} (R_{bdf}^p + 7 R_{bfd}^p) + R_{pcad} (R_{bef}^p + 7 R_{bfe}^p)) / 45 \\ & + (- (R_{pdbc} (5 R_{aef}^p - R_{afe}^p)) - R_{pdac} (5 R_{bef}^p - R_{bfe}^p) \\ & + 2 R_{pbae} (5 R_{cde}^p - R_{ced}^p) + 2 R_{pbae} (5 R_{cdf}^p - R_{cfd}^p) \\ & + 2 R_{pbad} (5 R_{cef}^p - R_{cfe}^p) - R_{pdab} (5 R_{cef}^p - R_{cfe}^p) \\ & + 2 R_{pbac} (5 R_{def}^p - R_{dfe}^p) - R_{pcab} (5 R_{def}^p - R_{dfe}^p)) / 45 \end{aligned} \quad (\text{A14})$$

APPENDIX B: END POINT SERIES EXPANSION

The basic input into the computation of the stress tensor or noise kernel is the Green function, a perfect example of a bi-scalar. We want to express it in such a way we can easily identify how it depends on the distance between its two support points. This leads us to consider *series expansions* of bi-scalars. The techniques are also useful for the series expansions of bi-tensors.

The world function σ introduced above provides the ideal geometric object for such a construction. It contains both distance and direction information. For a bi-scalar $S(x, y)$, the *end point expansion* is

$$S(x, y) = A^{(0)} + \sigma^p A_p^{(1)} + \sigma^p \sigma^q A_{pq}^{(2)} + \cdots + \sigma^{p_1} \cdots \sigma^{p_n} A_{p_1 \dots p_n}^{(n)} + \cdots, \quad (\text{B1})$$

so-called because the expansion tensors $A_{a_1 \dots a_n}^{(n)} = A_{a_1 \dots a_n}^{(n)}(x)$ have support at one of the end points for which $S(x, y)$ has support.

It is only the symmetric part of the expansion tensors $A_{a_1 \dots a_n}^{(n)}$ that contributes to the expansion, since they are contracted against symmetric products of σ^{p_i} 's. Moreover, the expansion tensors are order n in distance contribution to the bi-scalar $S(x, y)$. We also find it convenient to have an expansion where the distance dependence is separated from the direction dependence. To this end, if p^a is the unit vector along the geodesic from x to y , $\sigma^a = (2\sigma)^{\frac{1}{2}} p^a$, the expansion (B1) can be re-expressed as

$$S(x, y) = A^{(0)} + \sigma^{\frac{1}{2}} A^{(1)} + \sigma A^{(2)} + \dots + \sigma^{\frac{n}{2}} A^{(n)} + \dots \quad (\text{B2})$$

where $A^{(n)} = 2^{\frac{n}{2}} p^{p_1} \dots p^{p_n} A_{p_1 \dots p_n}^{(n)}$. Now the expansion scalars $A^{(n)}$ carry the direction information.

When multiplying series, this form readily collects terms by their order in distance. In the context of symbolic manipulation of series on the computer, this alternate form greatly improves processing speed.

The first series expansion to consider is that for the world function, which, by virtue of its defining differential equation is given by

$$\sigma(x, y) = \frac{g_{pq}}{2} \sigma^p \sigma^q \quad (\text{B3})$$

This is exact and from it we can see that all expansion tensors of order $n \geq 3$ vanish. This in turn tells us that $[\sigma; (a_1 a_2 a_3 \dots a_n)] = 0$, *i.e.*, the coincident limit of three or more symmetrized covariant derivatives of the world function vanish. This can also be seen by direct inspection of the previously given expressions for the coincident limits.

We now turn to relating the expansion tensors to the coincident limits of the derivatives of the scalar S . This is done by taking covariant derivatives and then the coincident limit of (B1). We immediately get

$$[S] = A^{(0)}. \quad (\text{B4})$$

Taking one derivative and the coincident limit gives

$$[S;_a] = A^{(1)}_a + A^{(0)}_{;a} \Rightarrow A^{(1)}_a = -[S]_{;a} + [S;_a]. \quad (\text{B5})$$

Two covariant derivatives yields

$$[S;_{ab}] = A^{(2)}_{ab} + A^{(2)}_{ba} + A^{(1)}_{a;b} + A^{(1)}_{b;a} + A^{(0)}_{;ab}. \quad (\text{B6})$$

We only need the symmetric part of $A^{(2)}_{ab}$, or if we assume $A^{(2)}_{ab}$ is symmetric, we solve for:

$$A^{(2)}_{ab} = \frac{1}{2} (-[S;_a]_{;b} - [S;_b]_{;a} + [S]_{;ab} + [S;_{ab}]) \quad (\text{B7})$$

We write this as

$$A^{(2)}_{ab} \doteq -[S;_a]_{;b} + \frac{[S]_{;ab}}{2} + \frac{[S;_{ab}]}{2} \quad (\text{B8})$$

where we use the standard notation \doteq to denote equality upon symmetrization. In terms of symbolic processing, this is implemented by taking each term of a tensorial expression and putting all free indices in lexicographic order. We also define rules that set to zero any Riemann curvature tensor R_{abcd} when either the first or second pair of indices are free. For example, consider when we take three covariant derivatives:

$$\begin{aligned} [S;_{abc}] &= A^{(3)}_{abc} + A^{(3)}_{acb} + A^{(3)}_{bac} + A^{(3)}_{bca} + A^{(3)}_{cab} + A^{(3)}_{cba} \\ &\quad + A^{(2)}_{ab;c} + A^{(2)}_{ac;b} + A^{(2)}_{ba;c} + A^{(2)}_{bc;a} + A^{(2)}_{ca;b} + A^{(2)}_{cb;a} \\ &\quad + A^{(1)}_{a;bc} + A^{(1)}_{b;ac} + A^{(1)}_{c;ab} + \frac{A^{(1)}_p R_{abc}{}^p}{3} + \frac{A^{(1)}_p R_{acb}{}^p}{3} \\ &\quad + A^{(0)}_{;abc} \end{aligned} \quad (\text{B9})$$

Now putting all free indices in lexicographic order and then using the above rules for the Riemann curvature tensor gives

$$\begin{aligned} [S;_{abc}] &\doteq 6 A^{(3)}_{abc} + 6 A^{(2)}_{ab;c} + 3 A^{(1)}_{a;bc} + A^{(0)}_{;abc} + \frac{2 A^{(1)}_p R_{abc}{}^p}{3} \\ &\doteq 6 A^{(3)}_{abc} + 6 A^{(2)}_{ab;c} + 3 A^{(1)}_{a;bc} + A^{(0)}_{;abc} \end{aligned} \quad (\text{B10})$$

This can now be solved for $A^{(3)}_{abc}$ and the previously determined results for $A^{(0)}$, $A^{(1)}_a$ and $A^{(2)}_{ab}$ used to get $A^{(3)}_{abc}$ solely in terms of the coincident limit of up to three covariant derivatives acting on S . Nothing new is encountered when determining the rest of the expansion tensors. We now give the results for up through $A^{(8)}_{abcdefgh}$:

$$A^{(3)}_{abc} \doteq \frac{1}{6}[S_{;abc}] - \frac{1}{2}[S_{;ab}]_{;c} + \frac{1}{2}[S_{;a}]_{;bc} - \frac{1}{6}[S]_{;abc} \quad (\text{B11a})$$

$$A^{(4)}_{abcd} \doteq \frac{1}{24}[S_{;abcd}] - \frac{1}{6}[S_{;abc}]_{;d} + \frac{1}{4}[S_{;ab}]_{;cd} - \frac{1}{6}[S_{;a}]_{;bcd} + \frac{1}{24}[S]_{;abcd} \quad (\text{B11b})$$

$$A^{(5)}_{abcde} \doteq \frac{1}{120}[S_{;abcde}] - \frac{1}{24}[S_{;abcd}]_{;e} + \frac{1}{12}[S_{;abc}]_{;de} - \frac{1}{12}[S_{;ab}]_{;cde} + \frac{1}{24}[S_{;a}]_{;bcde} - \frac{1}{120}[S]_{;abcde} \quad (\text{B11c})$$

$$A^{(6)}_{abcdef} \doteq \frac{1}{720}[S_{;abcdef}] - \frac{1}{120}[S_{;abcde}]_{;f} + \frac{1}{48}[S_{;abcd}]_{;ef} - \frac{1}{36}[S_{;abc}]_{;def} + \frac{1}{48}[S_{;ab}]_{;cdef} - \frac{1}{120}[S_{;a}]_{;bcdef} + \frac{1}{720}[S]_{;abcdef} \quad (\text{B11d})$$

$$A^{(7)}_{abcdefg} \doteq \frac{1}{5040}[S_{;abcdefg}] - \frac{1}{720}[S_{;abcdef}]_{;g} + \frac{1}{240}[S_{;abcde}]_{;fg} - \frac{1}{144}[S_{;abcd}]_{;efg} + \frac{1}{144}[S_{;abc}]_{;defg} - \frac{1}{240}[S_{;ab}]_{;cdefg} + \frac{1}{720}[S_{;a}]_{;bcdefg} - \frac{1}{5040}[S]_{;abcdefg} \quad (\text{B11e})$$

$$A^{(8)}_{abcdefgh} \doteq \frac{1}{40320}[S_{;abcdefgh}] - \frac{1}{5040}[S_{;abcdefg}]_{;h} + \frac{1}{1440}[S_{;abcdef}]_{;gh} - \frac{1}{720}[S_{;abcde}]_{;fgh} + \frac{1}{576}[S_{;abcd}]_{;efgh} - \frac{1}{720}[S_{;abc}]_{;defgh} + \frac{1}{1440}[S_{;ab}]_{;cdefgh} - \frac{1}{5040}[S_{;a}]_{;bcdefgh} + \frac{1}{40320}[S]_{;abcdefgh} \quad (\text{B11f})$$

These relations simplify considerable if the scalar S is symmetric, $S(x, y) = S(y, x)$, for the symmetrized odd derivatives are determined by the even derivatives. For one derivative,

$$S(x, y)_{;a'} = S(y, x)_{;a'} \quad \Rightarrow \quad [S_{;a'}] = [S_{;a}], \quad (\text{B12})$$

and applying Synge's theorem to the left-hand side above yields

$$[S]_{;a} - [S_{;a}] = [S_{;a}] \quad \Rightarrow \quad [S_{;a}] = \frac{[S]_{;a}}{2} \quad (\text{B13})$$

For three derivatives, we have

$$[S_{;a'b'c'}] = [S_{;abc}] \quad (\text{B14})$$

Once again, using Synge's theorem and (B13):

$$[S_{;ab}]_{;c} + [S_{;ac}]_{;b} + [S_{;bc}]_{;a} - \frac{[S]_{;acb}}{2} - [S_{;bca}] = [S_{;abc}] \quad (\text{B15})$$

It follows from this that

$$4 [S_{;(abc)}] = 6 [S_{;(ab)}]_{;c} - [S]_{;(abc)} \quad (\text{B16a})$$

The results for five and seven derivatives are

$$2 [S_{;(abcde)}] = 5 [S_{;(abcd)}]_{;e} - 5 [S_{;(ab)}]_{;cde} + [S]_{;(abcde)} \quad (\text{B16b})$$

$$8 [S_{;(abcdefg)}] = 28 [S_{;(abcdef)}]_{;g} - 70 [S_{;(abcd)}]_{;efg} + 84 [S_{;(ab)}]_{;cdefg} - 17 [S]_{;(abcdefg)} \quad (\text{B16c})$$

With these results, the equations for the even expansion tensors for a symmetric function simplify:

$$A^{(0)} = [S] \quad (B17a)$$

$$2!A^{(2)}_{ab} = [S;_{ab}] \quad (B17b)$$

$$4!A^{(4)}_{abcd} \doteq [S;_{abcd}] \quad (B17c)$$

$$6!A^{(6)}_{abcdef} \doteq [S;_{abcdef}] \quad (B17d)$$

$$8!A^{(8)}_{abcdefgh} \doteq [S;_{abcdefgh}] \quad (B17e)$$

while the odd expansion tensors are given in terms of the even tensors:

$$2!A^{(1)}_a \doteq -A^{(0)}_{;a} \quad (B18a)$$

$$4!A^{(3)}_{abc} \doteq -12 A^{(2)}_{ab;c} + A^{(0)}_{;abc} \quad (B18b)$$

$$6!A^{(5)}_{abcde} \doteq -360 A^{(4)}_{abcd;e} + 30 A^{(2)}_{ab;cde} - 3 A^{(0)}_{;abcde} \quad (B18c)$$

$$8!A^{(7)}_{abcdefg} \doteq -20160 A^{(6)}_{abcdef;g} + 1680 A^{(4)}_{abcd;efg} - 168 A^{(2)}_{ab;cdefg} + 17 A^{(0)}_{;abcdefg} \quad (B18d)$$

Very often, we need the covariant derivative of a series expansion (B1):

$$S_{;a} = A^{(0)}_{;a} + \sigma^{;p} A^{(1)}_{p;a} + A^{(1)}_p \sigma_{;a}^p + \sigma^{;p} \sigma^{;q} A^{(2)}_{pq;a} + A^{(2)}_{pq} \sigma^{;q} \sigma_{;a}^p + A^{(2)}_{pq} \sigma^{;p} \sigma_{;a}^q \dots \quad (B19)$$

If we replace $\sigma_{;ab}$ with its series expansion, then we readily have the series expansion of $S_{;a}$. We can get this via the above relations by merely replacing S with $\sigma_{;ab}$. In particular, we want the expansion

$$\begin{aligned} \sigma_{;ab} = & B^{(0)}_{ab} + B^{(1)}_{abp} \sigma^{;p} + B^{(2)}_{abpq} \sigma^{;p} \sigma^{;q} + B^{(3)}_{abpqr} \sigma^{;p} \sigma^{;q} \sigma^{;r} \\ & + B^{(4)}_{abpqrs} \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} + B^{(5)}_{abpqrst} \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} \sigma^{;t} \\ & + B^{(6)}_{abpqrst u} \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} \sigma^{;t} \sigma^{;u} \end{aligned} \quad (B20)$$

We immediately have

$$B^{(0)}_{ab} = [\sigma_{;ab}] = g_{ab} \quad (B21)$$

and

$$B^{(1)}_{abc} = -[\sigma_{;ab}]_{;c} + [\sigma_{;abc}] = 0 \quad (B22)$$

For the second order term:

$$\begin{aligned} B^{(2)}_{abcd} &= -[\sigma_{;abc}]_{;d} + \frac{[\sigma_{;ab}]_{;cd}}{2} + \frac{[\sigma_{;abcd}]}{2} \\ &= -\frac{1}{6} (R_{acbd} + R_{adbc}) \end{aligned} \quad (B23)$$

Now we have to be careful about how we carry out the symmetrization: it is only the indices c, d that are contracted over in the series (B20). So it is only the free indices *other than* a and b in this and the following that we put in lexicographic order (our routine for ordering free indices can be given a list of indices to exclude from ordering):

$$B^{(2)}_{abcd} \doteq' -\frac{R_{acbd}}{3} \quad (B24)$$

Equality upon symmetrization of all indices but a, b is denoted by \doteq' . The rest of the expansion tensors are computed in the same way; the results are

$$B^{(3)}_{abcde} \doteq' \frac{R_{acbd;e}}{12} \quad (B25a)$$

$$B^{(4)}_{abcdef} \doteq' \frac{1}{180} (-3 R_{acbd;ef} + 4 R_{pcad} R_{bef}{}^p) \quad (B25b)$$

$$B^{(5)}_{abcdefg} \doteq' \frac{1}{360} (R_{acbd;efg} - 3 R_{pdbc;e} R_{agf}{}^p - 3 R_{pdac;e} R_{bgf}{}^p) \quad (B25c)$$

$$\begin{aligned} B^{(6)}_{abcdefgh} \doteq' & \frac{1}{15120} (-219 R_{pdac;e} R_{bfg}{}^p{}_{;h} - 6 R_{acbd;efgh} \\ & - 114 R_{pdbc;ef} R_{agh}{}^p - 114 R_{pdac;ef} R_{bgh}{}^p \\ & + 24 R_{peaf} R_{qhb} R_c{}^p{}_d{}^q + 8 R_{peaf} R_{qhb} R_c{}^q{}_d{}^p) \end{aligned} \quad (B25d)$$

APPENDIX C: VANVLECK-MORETTE DETERMINANT

Other than the world function, the other main geometric object we need is the VanVlette-Morette determinant, defined as

$$D(x, y) \equiv -\det(-\sigma_{;ab'}) . \quad (C1)$$

In the context of Green function, what appears is

$$\Delta^{1/2}(x, y) = \left(\frac{D(x, y)}{\sqrt{g(x)g(y)}} \right)^{\frac{1}{2}} \quad (C2)$$

which we focus on. Using $2\sigma = \sigma^{;p}\sigma_{;p}$ the VanVlette-Morette determinant is seen to satisfy

$$D^{-1}(D\sigma^{;p})_{;p} = 4 \quad \Rightarrow \quad \Delta^{1/2}(4 - \square\sigma) - 2\Delta^{1/2}_{;p}\sigma^{;p} = 0 \quad (C3)$$

along with

$$[D] = g(x) \quad \Rightarrow \quad [\Delta^{1/2}] = 1 \quad (C4)$$

from which we readily get

$$[\Delta^{1/2}_{;a}] = 0. \quad (C5)$$

We could at this point proceed as we did with σ to determine the coincident limit expression of covariant derivatives of $\Delta^{1/2}$. But what we need is the end point expansion of $\Delta^{1/2}(x, y)$ to sixth order in σ^a . With this in mind, we set out to directly determine the series. We start by assuming the expansion

$$\begin{aligned} \Delta^{1/2} = & 1 + \Delta^{(2)}_{pq}\sigma^p\sigma^q + \Delta^{(3)}_{pqr}\sigma^p\sigma^q\sigma^r + \Delta^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s + \Delta^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t \\ & + \Delta^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u \end{aligned} \quad (C6)$$

and substitute back into (C3). We use (B20) and $\sigma^a = (2\sigma)^{\frac{1}{2}}p^a$. The expansion tensors $\Delta^{(n)}_{a_1\dots a_n}$ are determined by collecting terms according to their order in σ and setting to zero.

From (B20) and (B25d), we have the series expansion

$$\begin{aligned} \square\sigma = & 4 - \frac{2}{3}\sigma p^p p^q R_{pq} + \frac{\sqrt{2}}{6}\sigma^{\frac{3}{2}} R_{pq;r} p^p p^q p^r \\ & - \frac{1}{45}\sigma^2 p^p p^q p^r p^s (3 R_{pq;rs} + 4 R_{ptqu} R_r^u{}_s^t) \\ & + \frac{\sqrt{2}}{90}\sigma^{\frac{5}{2}} p^p p^q p^r p^s p^t (R_{pq;rst} + 6 R_{pvqu;r} R_s^v{}_t^u) \\ & + \frac{1}{1890}\sigma^3 p^p p^q p^r p^s p^t p^u (219 R_{pvqw;r} R_s^v{}_t^w{}_{;u} - 6 R_{pq;rstu} \\ & \quad + 24 R_{pvqw} R_r^w{}_{sx} R_t^v{}_u^x + 228 R_{pqvw;rs} R_t^w{}_u^v \\ & \quad + 8 R_{pvqw} R_r^w{}_{sx} R_t^x{}_u^v) \end{aligned} \quad (C7)$$

We have made the split into σ and p^a so we can readily carry out the needed multiplication in the series (C6), once we put it into the same form. The last piece we need is $\Delta^{1/2}_{;a}$. This is obtained by differentiating (C6) and then substituting (B20) for the $\sigma_{;ab}$ terms that arise. Once this is done, we have all the terms for the series expansion of (C3). The order σ term:

$$\frac{2\sigma p^p p^q}{3} (12 \Delta^{(2)}_{pq} - R_{pq}) = 0 \quad \Rightarrow \quad \Delta^{(2)}_{ab} = \frac{R_{ab}}{12} \quad (C8)$$

The order $\sigma^{\frac{3}{2}}$ term starts off as

$$\frac{\sigma^{\frac{3}{2}} p^p p^q p^r}{3\sqrt{2}} (24 \Delta^{(2)}_{pq;r} + R_{pq;r} + 72 \Delta^{(3)}_{pqr}) = 0$$

$$\Rightarrow \Delta^{(3)}_{abc} \doteq -\frac{1}{72} \left(24 \Delta^{(2)}_{ab;c} + R_{ab;c} \right). \quad (\text{C9})$$

Using (C8) shows

$$\Delta^{(3)}_{abc} \doteq -\frac{R_{ab;c}}{24} \quad (\text{C10})$$

For the order σ^2 term, we find

$$\begin{aligned} \Delta^{(4)}_{abcd} \doteq & \frac{1}{1440} \left(-360 \Delta^{(3)}_{abc;d} + 3 R_{ab;cd} + 60 \Delta^{(2)}_{ab} R_{cd} + 120 \Delta^{(2)}_{pa} R_{bcd}{}^p \right. \\ & \left. + 120 \Delta^{(2)}_{ap} R_b{}^p{}_{cd} + 4 R_{paqb} R_c{}^q{}_{d}{}^p \right) \end{aligned} \quad (\text{C11})$$

The fourth and fifth terms above vanish since we only need equality up to symmetrization. Using (C8) and (C10), the final form for this term becomes

$$\Delta^{(4)}_{abcd} \doteq \frac{1}{1440} (18 R_{ab;cd} + 5 R_{ab} R_{cd} + 4 R_{paqb} R_c{}^q{}_{d}{}^p) \quad (\text{C12a})$$

The last two coefficients are determined in exactly the same manner. The results are:

$$\Delta^{(5)}_{abcde} \doteq -\frac{1}{1440} (4 R_{ab;cde} + 5 R_{ab;c} R_{de} + 4 R_{paqb;c} R_d{}^q{}_{e}{}^p) \quad (\text{C12b})$$

$$\begin{aligned} \Delta^{(6)}_{abcdef} \doteq & \frac{1}{362880} (315 R_{ab;c} R_{de;f} + 180 R_{ab;cde} f + 378 R_{ab;cd} R_{ef} \\ & 35 R_{ab} R_{cd} R_{ef} + 84 R_{ab} R_{pcqd} R_e{}^q{}_{f}{}^p - 270 R_{paqb;c} R_d{}^q{}_{e}{}^p{}_{;f} \\ & - 288 R_{paqb;cd} R_e{}^q{}_{f}{}^p + 64 R_{paqb} R_{rdc}{}^q R_e{}^r{}_{f}{}^p) \end{aligned} \quad (\text{C12c})$$

APPENDIX D: SERIES EXPANSION FOR $\Delta\tau$

In this section we determine the expansion

$$\Delta\tau^2 = \sigma^{;p} \sigma^{;q} \delta\tau^{(2)}_{pq} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \delta\tau^{(3)}_{pqr} + \sigma^{;p} \sigma^{;q} \sigma^{;r} \sigma^{;s} \delta\tau^{(4)}_{pqrs}. \quad (\text{D1})$$

Since this is the expansion of a symmetric function, the expansion tensors are related the coincident limit of the covariant derivatives of $\Delta\tau^2$ via (B17e) and (B18d). We also use $[\Delta\tau] = 0$, justifying the series expansion starts at order two in σ^a . The expansion tensors are

$$\delta\tau^{(2)}_{ab} \doteq [\Delta\tau;_a] [\Delta\tau;_b] \quad (\text{D2a})$$

$$\delta\tau^{(3)}_{abc} \doteq -([\Delta\tau;_a]_{;b} [\Delta\tau;_c]) \quad (\text{D2b})$$

$$\delta\tau^{(4)}_{abcd} \doteq \frac{[\Delta\tau;_{ab}] [\Delta\tau;_{cd}]}{4} + \frac{[\Delta\tau;_a] [\Delta\tau;_{bcd}]}{3} \quad (\text{D2c})$$

To evaluate the covariant derivatives, we start with $\Delta\tau_{;a} = \Delta\tau_{,a} = \delta_a{}^\tau$. Also: $\Delta\tau_{,ab} = 0$ and $[\Delta\tau;_a] = \Delta\tau_{;a} \Rightarrow [\Delta\tau;_a]_{;b} = \Delta\tau_{;ab}$.

Turning to the computation of two covariant derivatives:

$$\begin{aligned} \Delta\tau_{;ab} &= \Delta\tau_{;a,b} - \Gamma^p_{ab} \Delta\tau_{;p} \\ &= \Delta\tau_{;a,b} - \Gamma^p_{ab} \Delta\tau_{;p} \\ &= -\Gamma^p_{ab} \delta_p{}^\tau \\ &= -\Gamma^\tau_{ab} \end{aligned} \quad (\text{D3})$$

and three covariant derivatives:

$$\Delta\tau_{;abc} = -\Gamma^\tau_{ab;c} \quad (\text{D4})$$

Using these results, the expansion tensors (D2c) become

$$\delta\tau^{(2)}_{ab} = \delta_a{}^\tau \delta_b{}^\tau \quad (\text{D5a})$$

$$\delta\tau^{(3)}_{abc} = \Gamma^\tau_{ab} \delta_c{}^\tau \quad (\text{D5b})$$

$$\delta\tau^{(4)}_{abcd} = \frac{1}{4} \Gamma^\tau_{ab} \Gamma^\tau_{cd} - \frac{1}{3} \Gamma^\tau_{ab;c} \delta_d{}^\tau \quad (\text{D5c})$$

APPENDIX E: SERIES EXPANSION OF HADAMARD FORM

For the regularization of the coincident limit of the noise kernel, we need the Hadamard form

$$S(x, y) = \frac{1}{(4\pi)^2} \left(\frac{2\Delta^{1/2}}{\sigma} + (v_0 + \sigma v_1 + \sigma^2 v_2) \log \sigma + (\sigma w_1 + \sigma^2 w_2) \right) \quad (\text{E1})$$

to fourth order in σ^a . We review the standard techniques for finding these expansions and present the results we use. The functions $v_n(x, y)$ and $w_n(x, y)$, $n \geq 1$ are determined by demanding

$$(\square - R/6) S(x, y) = 0. \quad (\text{E2})$$

The arbitrary function $w_0(x, y)$ is assumed to vanish. Working to fourth order, we proceed in the now familiar pattern of expanding each of the function in a series expansion and then solve for the expansion tensors by putting the expansion in the equations derived from (E2). Using the differential operators

$$H_n = \sigma^{ip} \nabla_p + \left((n-1) + \frac{1}{2} (\square \sigma) \right) \quad (\text{E3a})$$

$$K = \square - \frac{R}{6} \quad (\text{E3b})$$

we need to solve

$$v_0(x, y) = v_0^{(0)} + \sigma^p v_{0p}^{(1)} + \sigma^p \sigma^q v_{0pq}^{(2)} + \sigma^p \sigma^q \sigma^r v_{0pqr}^{(3)} + \sigma^p \sigma^q \sigma^r \sigma^s v_{0pqrs}^{(4)} \quad (\text{E4a})$$

$$H_0 v_0 = -K \Delta^{1/2} \quad (\text{E4b})$$

$$v_1(x, y) \approx v_1^{(0)} + \sigma^p v_{1p}^{(1)} + \sigma^p \sigma^q v_{1pq}^{(2)} \quad (\text{E5a})$$

$$2H_1 v_1 = -K v_0 \quad (\text{E5b})$$

$$v_2(x, y) \approx v_2^{(0)} \quad (\text{E6a})$$

$$4H_2 v_2 = -K v_1 \quad (\text{E6b})$$

along with

$$w_1(x, y) \approx w_1^{(0)} + \sigma^p w_{1p}^{(1)} + \sigma^p \sigma^q w_{1pq}^{(2)} \quad (\text{E7a})$$

$$2H_1 w_1 + 2H_2 v_1 = 0 \quad (\text{E7b})$$

$$w_2(x, y) \approx w_2^{(0)} \quad (\text{E8a})$$

$$4H_2 w_2 + 2H_4 v_2 = -K w_1 \quad (\text{E8b})$$

Proceeding as in Appendix C, we find

$$v_0^{(0)} = 0 \quad (\text{E9a})$$

$$v_{0a}^{(1)} = 0 \quad (\text{E9b})$$

$$v_{0ab}^{(2)} = (R_{;ab} - 3 R_{ab;p}{}^p + 4 R_{pa} R_b{}^p - 2 R_{pq} R_a{}^q{}_b{}^p + 2 R_{pqra} R_b{}^{rpq}) / 360 \quad (\text{E9c})$$

$$\begin{aligned} v_{0abc}^{(3)} = & (5 R_{;abc} - 14 R_{pa;bc}{}^p - 7 R_{ab;pc}{}^p + 5 R_{ab;p}{}^p{}_c + 8 R_{ab;cp}{}^p \\ & + 10 R_{pa;b} R_c{}^p + 15 R_{ab;p} R_c{}^p - 10 R_{paqb;c} R^{pq} \\ & - 24 R_{pq;a} R_b{}^q{}_c{}^p + 4 R_{pa;q} R_b{}^q{}_c{}^p - 8 R_{paqr;b} R_c{}^{pqr} \\ & + 2 R_{paqb;r} R_c{}^{pqr} - 2 R_{paqr;b} R_c{}^{qpr} + 2 R_{paqb;r} R_c{}^{qpr} \end{aligned}$$

$$+2 R_{paqr;b} R_c{}^{rpq})/1440 \quad (\text{E9d})$$

$$\begin{aligned} v_{abcd}^{(4)} = & (-3150 R_{pa;b} R_{cd}{}^i{}^p - 2205 R_{ab;p} R_{cd}{}^i{}^p - 720 R_{pa;b} R_c{}^p{}_{;d} \\ & + 11160 R_{pq;a} R_b{}^p{}_{;c}{}^q{}_{;d} - 3600 R_{pa;q} R_b{}^p{}_{;c}{}^q{}_{;d} + 270 R_{paqb;r} R_c{}^p{}_{;d}{}^q{}_{;r} \\ & + 2556 R_{pqra;b} R_c{}^p{}_{;d}{}^q{}_{;r} + 36 R_{pqra;b} R_c{}^{pqr}{}_{;d} - 522 R_{paqb;r} R_c{}^{pqr}{}_{;d} \\ & - 522 R_{paqb;r} R_c{}^{qpr}{}_{;d} + 468 R_{pqra;b} R_c{}^{rpq}{}_{;d} - 1170 R_{abcd} \\ & + 1656 R_{pa;bcd}{}^p + 1404 R_{pa;bc}{}^p{}_{;d} + 828 R_{ab;pcd}{}^p \\ & + 702 R_{ab;pc}{}^p{}_{;d} - 810 R_{ab;p}{}^p{}_{;cd} + 828 R_{ab;cpd}{}^p \\ & - 1188 R_{ab;cp}{}^p{}_{;d} - 1440 R_{ab;cdp}{}^p + 210 R_{ab} R_{cd} \\ & - 630 R_{ab;p}{}^p R_{cd} - 3996 R_{pa;bc} R_d{}^p - 4914 R_{ab;pc} R_d{}^p \\ & + 1008 R_{ab;cp} R_d{}^p + 840 R_{pa} R_{bc} R_d{}^p + 2916 R_{paqb;cd} R^{pq} \\ & - 336 R_{pa} R_{bcd}{}^p + 1008 R_{pa;q}{}^q R_{bcd}{}^p + 1344 R_{pq} R_a{}^q R_{bcd}{}^p \\ & - 336 R_{pa} R_{qrb}{}^p R_{cd}{}^{qr} + 10152 R_{pq;ab} R_c{}^p{}_{;d}{}^q + 2088 R_{pa;qb} R_c{}^p{}_{;d}{}^q \\ & - 4320 R_{pa;bq} R_c{}^p{}_{;d}{}^q - 1728 R_{ab;pq} R_c{}^p{}_{;d}{}^q + 672 R_{pa} R_{qb} R_c{}^p{}_{;d}{}^q \\ & - 420 R_{pq} R_{ab} R_c{}^p{}_{;d}{}^q + 4728 R_{pa} R_{qbr}{}^p R_c{}^q{}_{;d}{}^r + 1536 R_{pq} R_{rab}{}^p R_c{}^q{}_{;d}{}^r \\ & + 64 R_{pqra} R_{sb}{}^{pr} R_c{}^q{}_{;d}{}^s + 64 R_{pqra} R_s{}^p{}_{;b}{}^r R_c{}^q{}_{;d}{}^s + 3168 R_{pqra} R_s{}^r{}_{;b}{}^p R_c{}^q{}_{;d}{}^s \\ & - 1392 R_{pqra} R_{sb}{}^{pq} R_c{}^r{}_{;d}{}^s + 72 R_{pqra;bc} R_d{}^{pqr} - 540 R_{paqb;rc} R_d{}^{pqr} \\ & - 1296 R_{paqb;cr} R_d{}^{pqr} - 672 R_{pq} R_{rabc} R_d{}^{pqr} + 1344 R_{pqra} R_{sbc}{}^q R_d{}^{prs} \\ & - 540 R_{paqb;rc} R_d{}^{qpr} - 1296 R_{paqb;cr} R_d{}^{qpr} - 252 R_{pa} R_{qbrc} R_d{}^{qpr} \\ & + 432 R_{pqra;bc} R_d{}^{rpq} + 420 R_{ab} R_{pqrc} R_d{}^{rpq} - 252 R_{pa} R_{qbrc} R_d{}^{rpq} \\ & + 1344 R_{pqra} R_{sbc}{}^q R_d{}^{rps} - 128 R_{pqra} R_{sbc}{}^p R_d{}^{rqs} + 96 R_{pqra} R_{sbc}{}^r R_d{}^{spq} \\ & - 128 R_{pqra} R_{sbc}{}^p R_d{}^{sqr} - 672 R_{pqra} R_{sbc}{}^p R^{pqrs} - 1200 R_{paqb} R_{rcsd} R^{pqrs} \\ & - 624 R_{paqb} R_{rcsd} R^{psqr})/907200 \quad (\text{E9e}) \end{aligned}$$

$$v_1^{(0)} = (R_{;p}{}^p - R_{pq} R^{pq} + R_{pqrs} R^{pqrs})/360 \quad (\text{E10a})$$

$$\begin{aligned} v_{1a}^{(1)} = & (-6 R_{;pa}{}^p - 14 R_{;p}{}^p{}_a + 28 R_{pa;q}{}^{qp} - 22 R_{;p} R_a{}^p \\ & + 40 R_{pq;a} R^{pq} - 56 R_{pa;q} R^{pq} - 37 R_{pq;r} R_a{}^{pqr} - 19 R_{pq;r} R_a{}^{qpr} \\ & - 17 R_{pqrs;a} R^{pqrs} + 2 R_{pqrs;a} R^{psqr} + 12 R_{paqr;s} R^{psqr})/4320 \quad (\text{E10b}) \end{aligned}$$

$$\begin{aligned} v_{1ab}^{(2)} = & (210 R_{;p} R_{ab}{}^i{}^p + 380 R_{;p} R_a{}^p{}_{;b} - 360 R_{pa;q} R_b{}^p{}_{;q} - 2280 R_{pa;q} R_b{}^q{}_{;p} \\ & - 800 R_{pq;a} R^{pq}{}_{;b} - 200 R_{pa;q} R^{pq}{}_{;b} - 1488 R_{;pab}{}^p + 528 R_{;pa}{}^p{}_b \\ & + 820 R_{;p}{}^p{}_ab + 180 R_{;abp}{}^p + 960 R_{pq;ab}{}^{pq} + 960 R_{pa;qb}{}^{pq} \\ & + 960 R_{pa;qb}{}^{qp} - 1056 R_{pa;q}{}^q{}_b{}^p - 944 R_{pa;q}{}^{qp}{}_b + 960 R_{pa;bq}{}^{pq} \\ & - 1560 R_{pa;bq}{}^{qp} + 480 R_{ab;pq}{}^{pq} - 780 R_{ab;pq}{}^{qp} + 480 R_{ab;p}{}^p{}_q{}^q \\ & + 140 R_{;p}{}^p{}_ab - 152 R_{;pa}{}^p{}_b - 264 R_{pa;q}{}^q{}_b{}^p - 1232 R_{pq;ab} R^{pq} \\ & + 1768 R_{pa;qb} R^{pq} - 1056 R_{pa;bq} R^{pq} - 132 R_{ab;pq} R^{pq} \\ & - 3680 R_{pa} R_{qb} R^{pq} - 140 R_{pq} R_{ab} R^{pq} \\ & + 1320 R_{pq;r} R_a{}^p{}_{;b}{}^q{}_{;r} + 360 R_{pq;r} R_a{}^p{}_{;b}{}^r{}_{;q} + 5560 R_{pq;r} R_a{}^{pqr}{}_{;b} \\ & - 516 R_{pqra;s} R_b{}^{pqr}{}_{;s} + 180 R_{pqra;s} R_b{}^{prs}{}_{;q} + 12 R_{pqra;s} R_b{}^{rpq}{}_{;s} \\ & + 180 R_{pqra;s} R_b{}^{rps}{}_{;q} + 705 R_{pqrs;a} R^{pqrs}{}_{;b} - 1716 R_{pqra;s} R^{pqrs}{}_{;b} \\ & - 1008 R_{pqra;s} R^{pqr}{}_{;s}{}_{;b} + 544 R_{pq} R_a{}^p{}_{;b}{}^q - 72 R_{pq;r} R_a{}^p{}_{;b}{}^q \\ & + 1256 R_{pq} R_r{}^p R_a{}^q{}_{;b}{}^r + 6832 R_{pq;ra} R_b{}^{pqr} - 672 R_{pq;ar} R_b{}^{pqr} \\ & + 1728 R_{pa;q} R_b{}^{pqr} - 4712 R_{pq} R_{ra} R_b{}^{pqr} + 720 R_{pa;q} R_b{}^{qpr} \\ & - 246 R_{pq} R_{rsa}{}^p R_b{}^{qrs} - 784 R_{pq} R_{ras}{}^p R_b{}^{qrs} - 276 R_{pq} R_{rsa}{}^p R_b{}^{rqs} \\ & - 464 R_{pq} R_{ras}{}^p R_b{}^{rqs} + 1472 R_{pq} R_{ras}{}^p R_b{}^{sqr} + 664 R_{pqrs;ab} R^{pqrs} \\ & - 576 R_{pqra;sb} R^{pqrs} - 1416 R_{pqra;bs} R^{pqrs} + 140 R_{ab} R_{pqrs} R^{pqrs} \\ & - 1020 R_{pqra} R_{stb}{}^r R^{pqst} + 64 R_{pqrs;ab} R^{pqrs} - 96 R_{pqra;sb} R^{pqrs} \\ & - 432 R_{pqra;bs} R^{pqrs} - 192 R_{paqb;rs} R^{pqrs} - 2576 R_{pq} R_{rasb} R^{pqrs} \\ & - 432 R_{pqra} R_{stb}{}^q R^{prst} + 1624 R_{pa} R_{qrsb} R^{psqr} + 240 R_{pqra} R_{stb}{}^r R^{psqt} \\ & + 240 R_{pqra} R_{stb}{}^q R^{psrt} + 992 R_{pqrs} R_{tab}{}^q R^{ptrs} - 16 R_{pqra} R_{stb}{}^p R^{qrst} \\ & - 2068 R_{pqra} R_{sbt}{}^p R^{qsrt} - 1972 R_{pqra} R_{stb}{}^p R^{qtrs} \\ & - 2988 R_{pqra} R_{sbt}{}^p R^{qtrs})/604800 \quad (\text{E10c}) \end{aligned}$$

$$v_2^{(0)} = (139 R_{;p} R^{;p} - 354 R_{pq;r} R^{pq;r} + 732 R_{pq;r} R^{pr;q} + 114 R_{;p}{}^p{}_q{}^q)$$

$$\begin{aligned}
& -288 R_{pq;r}{}^{rqp} + 508 R_{;pq} R^{pq} - 84 R_{pq;r}{}^r R^{pq} \\
& + 484 R_{pq} R_r{}^p R^{qr} / 302400 \\
& + (405 R_{pqrs;t} R^{pqrs;t} + 810 R_{pqrs;t} R^{pqrt;s} + 288 R_{pq;rs} R^{prqs} \\
& - 5520 R_{pq} R_{rs} R^{prqs} - 2352 R_{pq} R_{rst}{}^p R^{qtrs} \\
& - 3520 R_{pqrs} R_t{}^p{}^r{}^u R^{qt su} - 640 R_{pqrs} R_{tu}{}^{pq} R^{rstu}) / 3628800
\end{aligned} \tag{E11}$$

$$w_1^{(0)} = -(R_{;p}{}^p - R_{pq} R^{pq} + R_{pqrs} R^{pqrs}) / 240 \tag{E12a}$$

$$\begin{aligned}
w_{1a}^{(1)} = & (12 R_{;pa}{}^p + 31 R_{;p}{}^p{}_a - 56 R_{pa;q}{}^{qp} + 44 R_{;p} R_a{}^p \\
& - 86 R_{pq;a} R^{pq} + 112 R_{pa;q} R^{pq} + 74 R_{pq;r} R_a{}^{pqr} + 38 R_{pq;r} R_a{}^{qp r} \\
& + 40 R_{pqrs;a} R^{pqrs} - 4 R_{pqrs;a} R^{psqr} - 24 R_{paqr;s} R^{psqr}) / 6480
\end{aligned} \tag{E12b}$$

$$\begin{aligned}
w_{1ab}^{(2)} = & (-630 R_{;p} R_{ab}{}^{;p} - 1756 R_{;p} R_a{}^p{}_b + 1080 R_{pa;q} R_b{}^{p;q} + 6840 R_{pa;q} R_b{}^{q;p} \\
& + 3856 R_{pq;a} R^{pq}{}_b - 968 R_{pa;q} R^{pq}{}_b + 4464 R_{;pab}{}^p - 1752 R_{;pa}{}^p{}_b \\
& - 3020 R_{;p}{}^p{}_ab - 540 R_{;abp}{}^p - 2880 R_{pq;ab}{}^{pq} - 2880 R_{pa;qb}{}^{pq} \\
& - 2880 R_{pa;qb}{}^{qp} + 3168 R_{pa;q}{}^q{}_b{}^p + 3616 R_{pa;q}{}^{qp}{}_b - 2880 R_{pa;bq}{}^{pq} \\
& + 4680 R_{pa;bq}{}^{qp} - 1440 R_{ab;pq}{}^{pq} + 2340 R_{ab;pq}{}^{qp} - 1440 R_{ab;p}{}^p{}_q{}^q \\
& - 504 R_{;p}{}^p{}_ab - 160 R_{;pa} R_b{}^p + 792 R_{pa;q}{}^q{}_b{}^p + 5152 R_{pq;ab} R^{pq} \\
& - 6872 R_{pa;qb} R^{pq} + 3168 R_{pa;bq} R^{pq} + 396 R_{ab;pq} R^{pq} \\
& + 11040 R_{pa} R_{qb} R^{pq} + 504 R_{pq} R_{ab} R^{pq} \\
& - 3960 R_{pq;r} R_a{}^p{}_b{}^{q;r} - 1080 R_{pq;r} R_a{}^p{}_b{}^{r;q} - 18248 R_{pq;r} R_a{}^{pq r}{}_b \\
& + 1548 R_{pqra;s} R_b{}^{pq r}{}_s - 540 R_{pqra;s} R_b{}^{pr s}{}_q - 36 R_{pqra;s} R_b{}^{rpq}{}_s \\
& - 540 R_{pqra;s} R_b{}^{rps}{}_q - 2955 R_{pqrs;a} R^{pqrs}{}_b + 5484 R_{pqra;s} R^{pqrs}{}_b \\
& + 3024 R_{pqra;s} R^{prqs}{}_b - 1632 R_{;pq} R_a{}^p{}_b{}^q + 216 R_{pq;r}{}^r R_a{}^p{}_b{}^q \\
& - 3768 R_{pq} R_r{}^p R_a{}^q{}_b{}^r - 22064 R_{pq;ra} R_b{}^{pq r} + 2016 R_{pq;ar} R_b{}^{pq r} \\
& - 5184 R_{pa;q r} R_b{}^{pq r} + 14136 R_{pq} R_{ra} R_b{}^{pq r} - 2160 R_{pa;q r} R_b{}^{qp r} \\
& + 738 R_{pq} R_{rsa}{}^p R_b{}^{qrs} + 2352 R_{pq} R_{ras}{}^p R_b{}^{qrs} + 828 R_{pq} R_{rsa}{}^p R_b{}^{rqs} \\
& + 1392 R_{pq} R_{ras}{}^p R_b{}^{rqs} - 4416 R_{pq} R_{ras}{}^p R_b{}^{sq r} - 2804 R_{pqrs;ab} R^{pqrs} \\
& + 2064 R_{pqra;sb} R^{pqrs} + 4248 R_{pqra;bs} R^{pqrs} - 504 R_{ab} R_{pqrs} R^{pqrs} \\
& + 3060 R_{pqra} R_{stb}{}^r R^{pqst} - 248 R_{pqrs;ab} R^{prqs} + 288 R_{pqra;sb} R^{prqs} \\
& + 1296 R_{pqra;bs} R^{prqs} + 576 R_{paqb;rs} R^{prqs} + 7728 R_{pq} R_{rasb} R^{prqs} \\
& + 1296 R_{pqra} R_{stb}{}^q R^{prst} - 4872 R_{pa} R_{qrsb} R^{psqr} - 720 R_{pqra} R_{stb}{}^r R^{psqt} \\
& - 720 R_{pqra} R_{stb}{}^q R^{psrt} - 2976 R_{pqrs} R_{tab}{}^q R^{ptrs} + 48 R_{pqra} R_{stb}{}^p R^{qrst} \\
& + 6204 R_{pqra} R_{sbt}{}^p R^{qsrt} + 5916 R_{pqra} R_{stb}{}^p R^{qtrs} \\
& + 8964 R_{pqra} R_{sbt}{}^p R^{qtrs}) / 1451520
\end{aligned} \tag{E12c}$$

$$\begin{aligned}
w_2^{(0)} = & -(139 R_{;p} R^{;p} - 354 R_{pq;r} R^{pq;r} + 732 R_{pq;r} R^{pr;q} \\
& + 114 R_{;p}{}^p{}_q - 288 R_{pq;r}{}^{rqp} + 508 R_{;pq} R^{pq} \\
& - 84 R_{pq;r}{}^r R^{pq} + 484 R_{pq} R_r{}^p R^{qr}) / 145152 \\
& - (405 R_{pqrs;t} R^{pqrs;t} + 810 R_{pqrs;t} R^{pqrt;s} + 288 R_{pq;rs} R^{prqs} \\
& - 5520 R_{pq} R_{rs} R^{prqs} - 2352 R_{pq} R_{rst}{}^p R^{qtrs} - 3520 R_{pqrs} R_t{}^p{}^r{}^u R^{qt su} \\
& - 640 R_{pqrs} R_{tu}{}^{pq} R^{rstu}) / 1741824
\end{aligned} \tag{E13}$$

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- [36] We must stress, as we did in Paper I, that though all these are done on a computer, no numerical approximation is used. All work is done symbolically in terms of the explicit functional form of the metric and the parameters of the field. The final results are exact to the extent that the analytic form of the Green function is exact.
- [37] Since this work for analyzing the coincident limit of the noise kernel via the point separation method is tailored to symbolic computation on the computer, our method for deriving particular results is designed to take maximum advantage of the computer.